MATH 2050C Lecture 21 (Apr 8)

* Take-home Final: May 4 @ 6:00 PM – May 5 @ 6:00 PM. *

[Problem Set 11 posted, due on Apr 16.]

Three important theorems about continuous $f:[a,b]\to\mathbb{R}$

- Boundedness Thm
- Extreme Value Theorem
- Intermediate Value Theorem

[compactness]

Extreme Value Thm: A cts $f:[a,b]\to\mathbb{R}$ always achieve its absolute maximum and minimum, i.e.

$$\exists\, x^* \in [a,b] \text{ s.t. } f(x^*) = M := \sup\{f(x) \mid x\in[a,b]\}$$

$$\exists\, x_* \in [a,b] \text{ s.t. } f(x_*) = m := \inf\{f(x) \mid x\in[a,b]\}$$

not nec. unique

Proof: We only prove the existence of x^*.

Since $M := \sup\{f(x) \mid x\in[a,b]\}$, $\forall \varepsilon>0$, $\exists\, x_\varepsilon \in [a,b]$ st.

$$M-\varepsilon < f(x_\varepsilon)$$

Take $\varepsilon = \frac{1}{n}$, then we obtain a sequence $(x_n)\subseteq[a,b]$ st.

$$M-\frac{1}{n} < f(x_n) \leq M$$

By Bolzano-Weierstrass Thm, since (x_n) is a bdd seq.

$\Rightarrow \exists$ convergent subseq. (x_{n_k}) of (x_n), say $\underset{[a,b]}{\underset{\uparrow}{x^*}} := \lim(x_{n_k})$

Claim: $f(x^*) = M$

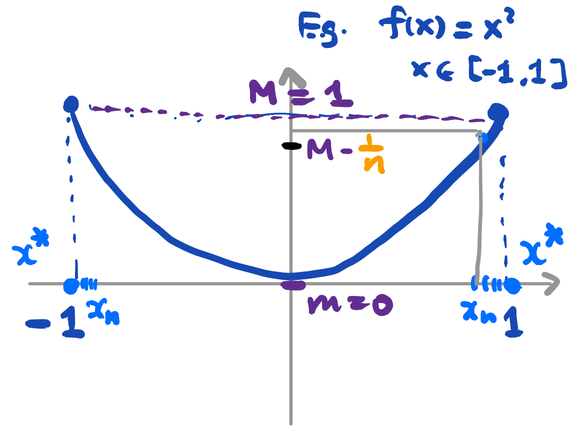
Pf: Since $M - \frac{1}{n_k} < f(x_{n_k}) \leq M$

for all $k \in \mathbb{N}$,

take $k \to \infty$, by continuity of f at x^*

$$M \leq f(x^*) = \lim_{k\to\infty} f(x_{n_k}) \leq M$$

Limit theorems



$\square$

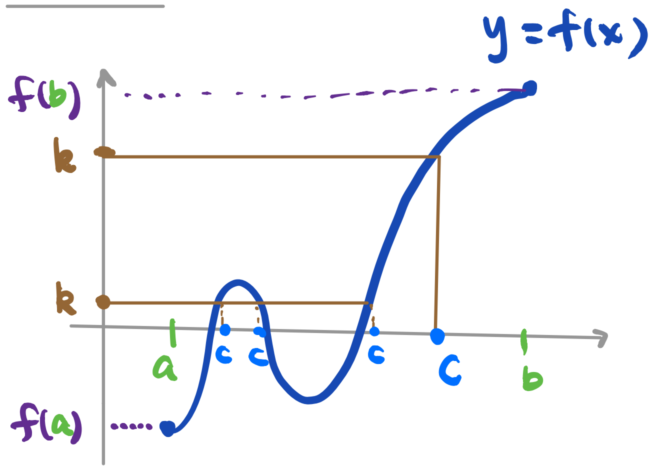
Intermediate Value Theorem [connectedness]

Let $f : [a,b] \to \mathbb{R}$ be a cts function s.t. $f(a) < f(b)$.

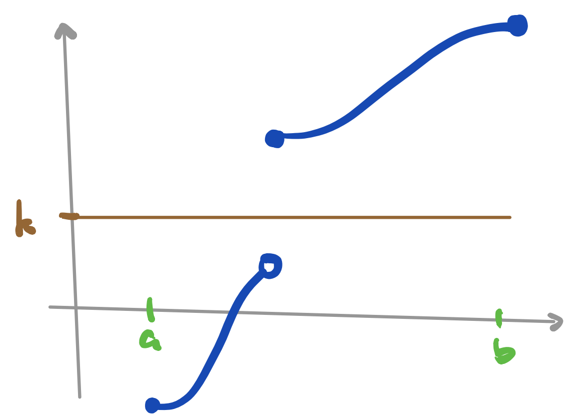
THEN, $\forall\, k \in (f(a), f(b))$, $\exists\, c \in [a,b]$ s.t.

$$f(c) = k$$

Picture:

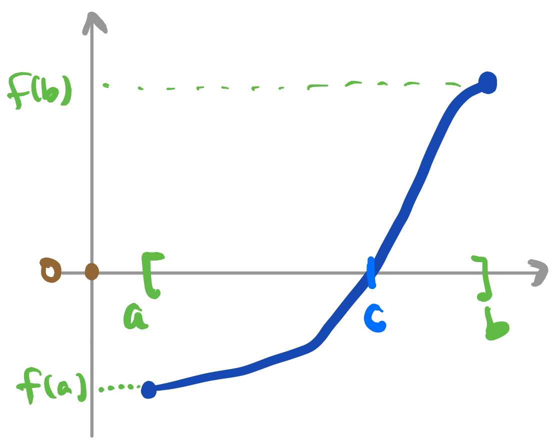


Continuity is needed



<u>Proof</u>: It suffices to consider the case:

$$f(a) < 0 < f(b) \quad \text{and} \quad k = 0$$



[$\because$ The general case follows by considering $g(x) := f(x) - k$.]

$g(c) = 0 \iff f(c) = k$

<u>Q</u>: How to locate a "root" where $f(c) = 0$?

<u>A</u>: Method of bisection.

Define a nested seq of closed & bdd intervals I_n as follows.

Take $I_1 := [a, b] =: [a_1, b_1]$

Consider the midpt. $\frac{a_1+b_1}{2}$ of I_1

<u>Case 1</u>: $f\left(\frac{a_1+b_1}{2}\right) < 0 \Rightarrow$ take $I_2 := [a_2, b_2] = \left[\frac{a_1+b_1}{2}, b_1\right]$

<u>Case 2</u>: $f\left(\frac{a_1+b_1}{2}\right) > 0 \Rightarrow$ take $I_2 := [a_2, b_2] = \left[a_1, \frac{a_1+b_1}{2}\right]$

<u>Case 3</u>: $f\left(\frac{a_1+b_1}{2}\right) = 0 \Rightarrow$ DONE, take $c = \frac{a_1+b_1}{2}$.

Repeat this process for I_2.

Either you locate a root (Case 3), or you obtain a seq. of closed & bdd intervals $I_n := [a_n, b_n]$. $\text{Length}(I_{n+1}) = \frac{1}{2}\text{Length}(I_n)$

st $\begin{cases} I_{n+1} \subseteq I_n & \forall n \in \mathbb{N} \quad \underline{\text{nested}} \\ f(a_n) < 0 < f(b_n) & \forall n \in \mathbb{N}. \quad \text{———} (\#) \end{cases}$

By Nested Interval Property, $\bigcap_{n=1}^{\infty} I_n = \{c\}$

($\because$ $\text{Length}(I_n) \to 0$)

Claim: $f(c) = 0$.

Pf: Since $\lim(a_n) = \lim(b_n) = c$, take $n \to \infty$ in (*),
by continuity of f at c,

$$f(c) \leq 0 \leq f(c), \text{ ie } f(c) = 0$$

$\square$